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CONDITIONS FOR EXTENSION OF SOME CLASSES OF SQUARE-SUMMABLE ANALYTIC FUNCTIONS

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ABSTRACT

Analogously to S. Bernstein's conditions for the possibility of extension of an analytic function from a given domain to a greater one, we give necessary and sufficient conditions for the extension of analytic functions submitted to different conditions of square-summability. The conditions are given by the characteristic of the rate of approximation of the given function by simple or generalized polynomials. For illustration we formulate one of the theorems which is proved in the paper: *Let $H^2_\sigma(C_r)$ be the class of analytic functions $f(z)$ in the disc $C_r = \{z : |z| < r\}$ for which*

$$\|f(z)\|^2_{H^2_\sigma(C_r)} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^r |f(\rho e^{i\theta})|^2 \sigma\left(\frac{\rho}{r}R\right) d\rho d\theta < \infty, \quad r < R,$$

where $\sigma(r)$ is a non-negative continuous and non-increasing function on $[0, R]$; $\sigma(r) > 0$ for $0 \leqq r < R$. Then $f(z) \in H^2_\sigma(C_R)$ if and only if $f(z) \in H^2_\sigma(C_r)$ and for every integer $n \geqq 0$ there exists a polynomial $p_n(z)$ of degree n at most such that

$$\|f(z) - p_n(z)\|_{H^2_\sigma(C_r)} \leqq \left(\frac{r}{R}\right)^n \cdot \beta_n \quad (n \geqq 0),$$

where $\beta_n \geqq 0$, $\sum_{n=0}^{\infty} \beta_n^2 < \infty$.

The purpose of this article is to examine the conditions for the extension of square-summable analytic functions from a given domain, on which they are defined, to a greater one. These conditions can be characterized by the rates of approximation of the function by smaller classes of functions, and in particular cases, by polynomials.

As an example for such conditions, in the case of analytic extension, we can take Bernstein's theorem ([3], p. 114): Let K be a continuum containing more than one point, and let K_∞ be the component of K^c — the complement of K in

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Riemann's sphere — containing the point at infinity. Let γ_R be the image of the circle $|w| = R > \rho$ under the conformal mapping

$$z = \psi(w) = w + \beta_0 + \frac{\beta_{-1}}{w} + \frac{\beta_{-2}}{w^2} + \cdots \tag{1}$$

of the domain $|w| > \rho$ onto K_∞. Let $f(z)$ be a function defined on K. Then $f(z)$ is analytic on the bounded domain enclosed by γ_R, denoted by $\mathrm{int}(\gamma_R)$, if and only if for any $\varepsilon > 0$ and any integer $n \geqq 0$, there exists a polynomial $p_n(z)$, of degree n at most, such that

$$\max_{z\in K} |f(z) - p_n(z)| < C(\varepsilon)(q+\varepsilon)^n \qquad (n = 0, 1, \cdots), \quad q < 1 \tag{2}$$

where $R = \rho/q$, and $C(\varepsilon)$ is a constant. In this case, the sequence $\{p_n(z)\}_0^\infty$ converges uniformly to $f(z)$ inside $\mathrm{int}(\gamma_R)$.

Let γ be a Jordan curve, r a positive number, and $\alpha = \{\alpha_i\}_0^\infty$ a positive sequence. We introduce the following notation:

$\mathrm{int}(\gamma)$ — the bounded domain enclosed by γ.

$\mathrm{ext}(\gamma)$ — the complementary domain to $\overline{\mathrm{int}(\gamma)}$, which is the closure of $\mathrm{int}(\gamma)$.

$\mathcal{A}_+(\gamma)$ — the space of analytic functions on $\mathrm{int}(\gamma)$.

$\mathcal{A}_-(\gamma)$ — the space of analytic functions on $\mathrm{ext}(\gamma)$.

$\overline{\mathcal{A}_+}(\gamma)$ — the space of analytic functions on $\overline{\mathrm{int}(\gamma)}$.

$\overline{\mathcal{A}_-}(\gamma)$ — the space of analytic functions on $\overline{\mathrm{ext}(\gamma)}$.

(For the definitions and topologies of the spaces of analytic functions, see [2].)

$l_\alpha^p(r)$ — the Banach space of all complex sequences $x = \{x_k\}_0^\infty$, such that

$$\|x\|_{l_\alpha^p(r)} = \sqrt[p]{\sum_{k=0}^{\infty} |x_k|^p r^{pk} \alpha_k} < \infty, \qquad 1 \leqq p \leqq \infty,$$

where $l_\alpha^p(1) = l^p$ when $\alpha_k = 1$ $(\forall k)$.

c_r — the circle $|w| = r$.

$E_+^2(\gamma)$ — the Banach space of functions $f(z) \in \mathcal{A}_+(\gamma)$ with the norm

$$\|f(z)\|_{E_+^2(\gamma)}^2 = \sup_n \int_{\gamma_n} |f(z)|^2 |dz| < \infty,$$

where $\{\gamma_n\}_0^\infty$ is a sequence of rectifiable Jordan curves in $\mathrm{int}(\gamma)$ which converges to γ.

$E_-^2(\gamma)$ — the Banach space of functions $f(z) \in \mathcal{A}_-(\gamma)$, with the norm

$$\|f(z)\|_{E_-^2(\gamma)}^2 = \sup_n \int_{\gamma_n} |f(z)|^2 |dz| < \infty,$$

where $\{\gamma_n\}_0^\infty$ is a sequence of rectifiable Jordan curves in $\operatorname{ext}(\gamma)$, which converges to γ.

F_n — the set of all complex sequences $x = \{x_k\}_0^\infty$, where $x_k = 0$ for $k > n$.

THEOREM 1. *The sequence $x = \{x_k\}_0^\infty$ belongs to $l_\alpha^p(R)$ if and only if $x \in l_\alpha^p(r)$ for $r < R$, and for every integer $n \geqq 0$, there exists $p^{(n)} \in F_n$ such that*

$$\|x - p^{(n)}\|_{l_\alpha^p(r)} \leqq \left(\frac{r}{R}\right)^n \beta_n \qquad (n = 0, 1, 2, \cdots)$$

where $\beta = \{\beta_k\}_0^\infty \in l^p$.

PROOF. Let $x = \{x_k\}_0^\infty \in l_\alpha^p(R)$, $r < R$, and $p^{(n)} = \{x_0, x_1, \cdots, x_n, 0, 0, \cdots\}$. Then

$$\|x - p^{(n)}\|_{l_\alpha^p(r)}^p = \sum_{k=n+1}^{\infty} |x_k|^p r^{pk} \alpha_k = \left(\frac{r}{R}\right)^{pn} \beta_n^p,$$

where $\beta_n^p = (R/r)^{pn} \sum_{k=n+1}^\infty |x_k|^p r^{pk} \alpha_k$, and therefore

$$\sum_{n=0}^{\infty} \beta_n^p = \sum_{n=0}^{\infty} \sum_{k=n+1}^{\infty} \left(\frac{R}{r}\right)^{pn} |x_k|^p r^{pk} \alpha_k = \sum_{k=1}^{\infty} r^{pk} |x_k|^p \alpha_k \sum_{n=0}^{k-1} \left(\frac{R}{r}\right)^{pn}$$

$$= \sum_{k=1}^{\infty} |x_k|^p \alpha_k \frac{R^{pk} - r^{pk}}{R^p - r^p} \cdot r^p < \infty.$$

Now, let $x \in l_\alpha^p(r)$, and $\|x - p^{(n)}\|_{l_\alpha^p(r)} \leqq (r/R)^n \beta_n$ $(n = 0, 1, 3, \cdots)$ where $\{\beta_k\}_0^\infty \in l^p$ and $p^{(n)} \in F_n$,

$$\|x - \{x_0, x_1, \cdots, x_n, 0, 0, \cdots\}\|_{l_\alpha^p(r)} \leqq \|x - p^{(n)}\|_{l_\alpha^p(r)} \leqq \left(\frac{r}{R}\right)^n \beta_n.$$

Then

$$\|\{x_0, \cdots, x_n, 0, \cdots\} - \{x_0, \cdots, x_{n-1}, 0, \cdots\}\|_{l_\alpha^p(r)} = |x_n| r^n \alpha_n^{1/p}$$

$$\leqq \|\{x_0, \cdots, x_n, 0, \cdots\} - x\|_{l_\alpha^p(r)} + \|x - \{x_0, \cdots, x_{n-1}, 0, \cdots\}\|_{l_\alpha^p(r)}$$

$$\leqq \left(\frac{r}{R}\right)^n \beta_n + \left(\frac{r}{R}\right)^{n-1} \beta_{n-1}$$

and therefore $|x_n| R^n \alpha_n^{1/p} \leqq \beta_n + (R/r)\beta_{n-1}$, so that $\{x_k R^k \alpha_k^{1/p}\}_{k=0}^\infty \in l^p$, which means that $\{x_k\}_0^\infty \in l_\alpha^p(R)$.

LEMMA. *The function $f(z) = \sum_{k=0}^\infty x_k z^k$ belongs to $\mathscr{A}_+(c_R)$ for every $\{x_k\}_0^\infty \in l_\alpha^p(R)$, $p \geqq 1$ if and only if $\sum_{k=0}^\infty \alpha_k^{-1} z^k \in \mathscr{A}_+(c_1)$, which is equivalent to the condition $\underline{\lim} \sqrt[k]{\alpha_k} \geqq 1$.*

PROOF. If $\sum_{k=0}^{\infty} \alpha_k^{-1} z^k \in \mathscr{A}_+(c_1)$, and $x \in l_\alpha^p(R)$ for $p > 1$, then for $\rho < R$

$$\sum_{k=0}^{\infty} |x_k| \rho^k \leqq \sqrt[p]{\sum_{k=0}^{\infty} |x_k|^p R^{pk} \alpha_k} \cdot \sqrt[q]{\sum_{k=0}^{\infty} \left(\frac{\rho}{R}\right)^{kq} \alpha_k^{1/(1-p)}} < \infty,$$

by Hölder's inequality. If $p = 1$, then, for some $c > 0$,

$$\sum_{k=0}^{\infty} |x_k| \rho^k = \sum_{k=0}^{\infty} |x_k| R^k \alpha_k \cdot \left(\frac{\rho}{R}\right)^k \frac{1}{\alpha_k} \leqq C \sum_{k=0}^{\infty} |x_k| R^k \alpha_k < \infty.$$

Conversely, if $\underline{\lim} \sqrt[k]{\alpha_k} < 1$ then there exists a subsequence $\{\alpha_{k_n}\}$, such that $\sqrt[k_n]{\alpha_{k_n}} \leqq \theta < 1$ $(n = 0, 1, 2, \cdots)$. We take $\eta < R$ such that $(R/\eta)^p \theta < 1$, and

$$x_k = \begin{cases} \eta^{-k} & \text{if } k = k_n, \\ 0 & \text{otherwise.} \end{cases}$$

Now, $\sum_{k=0}^{\infty} x_k \eta^k = \sum_{n=0}^{\infty} x_{k_n} \eta^{k_n} = \infty$, even though

$$\sum_{k=0}^{\infty} |x_k|^p R^{pk} \alpha_k = \sum_{n=0}^{\infty} |x_{k_n}|^p R^{pk_n} \alpha_{k_n} \leqq \sum_{n=0}^{\infty} \frac{1}{\eta^{pk_n}} R^{pk_n} \theta^{k_n} = \sum_{n=0}^{\infty} \left[\left(\frac{R}{\eta}\right)^p \theta\right]^{k_n} < \infty.$$

Let X and Y be two Banach spaces, and assume Y is algebraically and topologically contained in X. Furthermore, assume that there exists a linear continuous operator T which maps X onto $l_\alpha^p(r)$, and its restriction to Y maps Y onto $l_\alpha^p(R)$ continuously, where $r < R$. Let $y_n = T^{-1} e^{(n)}$, where $e^{(n)} = \{\delta_{nk}\}_{k=0}^{\infty}$ ($\{\delta_{nk}\}$ is Kronecker's δ) and let Y_n be the linear subspace spanned by $\{y_0, y_1, \cdots, y_n\}$. Clearly we can state

THEOREM 2. *$y \in Y$ if and only if $y \in X$ and for every integer $n \geqq 0$ there exists $p_n \in Y_n$ such that*

$$\| y - p_n \|_X \leqq \left(\frac{r}{R}\right)^n \beta_n$$

where $\{\beta_n\}_0^\infty \in l^p$.

Let us consider some applications:

(1) As we know ([1], [4]), Riesz's space $H_+^2(c_R)$ is the space of all functions $f(z) = \sum_{k=0}^{\infty} a_k z^k \in \mathscr{A}_+(c_R)$, for which

$$\| f(z) \|_{H_+^2(c_R)}^2 = \sup_{\rho < R} \frac{1}{2\pi} \int_0^{2\pi} |f(\rho e^{i\theta})|^2 d\theta = \sum_{k=0}^{\infty} |a_k|^2 R^{2k}$$

$$= \frac{1}{2\pi} \int_0^{2\pi} |f(Re^{i\theta})|^2 d\theta < \infty$$

where $f(Re^{i\theta})$ are the boundary values of $f(z)$ which exist almost everywhere and are square-summable.

The operator $Tf(z) = T\sum_{k=0}^{\infty} a_k z^k = \{a_k\}_0^\infty$ is an isometric isomorphism of $H_+^2(c_r)$ onto $l_\alpha^2(r)$, where $\alpha = \{1, 1, \cdots\}$, and its restriction to $H_+^2(c_R)$ conserves this property for the couple $H_+^2(c_R)$ and $l_\alpha^2(R)$, for $R > r$. Based on Theorem 2, we now have the following theorem:

THEOREM 3. *$f(z) \in H_+^2(c_R)$ if and only if $f(z) \in H_+^2(c_r)$, $r < R$, and for every integer $n \geqq 0$ there exists a polynomial $p_n(z)$ of degree at most n, such that*

$$\|f(z) - p_n(z)\|_{H_+^2(c_r)} \leqq \left(\frac{r}{R}\right)^n \beta_n \qquad (n = 0, 2, \cdots),$$

where $\{\beta_k\}_0^\infty \in l^2$.

(2) Let $\sigma(r)$ be a non-negative, continuous and non-increasing function defined on $[0, R]$, with $\sigma(r) > 0$ for $0 \leqq r < R$. We define the space

$$H_\sigma^2(c_r) = \left\{ f(z) : f(z) \in \mathscr{A}_+(c_r), \|f(z)\|_{H_\sigma^2(c_r)}^2 \right.$$
$$\left. = \frac{1}{2\pi} \int_0^{2\pi} \int_0^r |f(\rho e^{i\theta})|^2 \sigma\left(\frac{\rho}{r} R\right) d\rho d\theta < \infty \right\}.$$

Clearly, $\|f(z)\|_{H_\sigma^2(c_r)}^2 = \|\sum_k a_k z^k\|_{H_\sigma^2(c_r)}^2 = r\sum_{k=0}^{\infty} |a_k|^2 r^{2k} \alpha_k = r\|\{a_k\}\|_{l_\alpha^2(r)}^2$ where $\alpha = \{\alpha_k\}_0^\infty$ and

(3) $$\alpha_k = \frac{1}{R^{2k+1}} \int_0^R \rho^{2k} \sigma(\rho) d\rho.$$

It follows from (3) that for $0 < \theta < 1$

$$\alpha_k \geqq \frac{1}{R^{2k+1}} \int_0^{R\theta} \rho^{2k} \sigma(R\theta) d\rho = \sigma(R\theta) \frac{\theta^{2k+1}}{2k+1}.$$

Therefore $\underline{\lim}_{k\to\infty} \sqrt[k]{\alpha_k} \geqq \theta^2$, hence $\underline{\lim} \sqrt[k]{\alpha_k} \geqq 1$.

The operator $Tf(z) = T\sum_{k=0}^{\infty} a_k z^k = \{a_k\}_0^\infty$ is a one-to-one linear mapping of $H_\sigma^2(c_r)$ onto $l_\alpha^2(r)$ for every r, $0 < r \leqq R$. Based on Theorem 2, we have the following theorem:

THEOREM 4. *$f(z) \in H_\sigma^2(c_R)$ if and only if $f(z) \in H_\sigma^2(c_r)$, $r < R$, and, for every integer $n \geqq 0$, there exists a polynomial $p_n(z)$ of degree at most n such that*

$$\|f(z) - p_n(z)\|_{H_\sigma^2(c_r)} \leqq \left(\frac{r}{R}\right)^n \cdot \beta_n \qquad (n = 0, 1, 2, \cdots),$$

where $\{\beta_k\}_0^\infty \in l^2$.

(3) Consider $A^{-\infty}(r)$ — the class of functions $f(z) \in \mathcal{A}_+(c_r)$, such that

$$|f(z)| \leqq \frac{c}{(r-|z|)^m}, \tag{4}$$

where the constant c, $c > 0$ and the natural number m depends on f.

THEOREM 5. *$f(z) \in A^{-\infty}(R)$ if and only if $f(z) \in A^{-\infty}(r)$, $r < R$, and for some $\alpha > 0$, there exists, for every integer $n \geqq 0$, a polynomial $p_n(z)$ of degree at most n, such that*

$$\int_0^{2\pi}\int_0^r (r-\rho)^{\alpha-1}|f(\rho e^{i\theta}) - p_n(\rho e^{i\theta})|^2 d\rho d\theta \leqq \left(\frac{r}{R}\right)^n \cdot \beta_n$$

for some $\beta = \{\beta_k\}_0^\infty \in l^2$.

PROOF. Let s be a positive number. Consider the space $H_s^2(c_r)$ of all $f(z) \in \mathcal{A}_+(c_r)$ such that

$$\|f(z)\|^2_{H_s^2(c_r)} = \frac{1}{2\pi}\int_0^{2\pi}\int_0^r (r-\rho)^{s-1}|f(\rho e^{i\theta})|^2 d\rho d\theta < \infty.$$

We argue that $\bigcup_{s>0} H_s^2(c_r) = A^{-\infty}(r)$. If $f(z) \in A^{-\infty}(r)$ and $f(z)$ behaves as in (4), then obviously $f(z)$ belongs to $H_s^2(c_r)$ for some s, great enough. Conversely, let $f(z) = \sum_{k=0}^\infty a_k z^k \in H_s^2(c_r)$. Then $f(rz) \in H_s^2(c_1)$. If we put $f(rz) = \sum_{k=0}^\infty b_k z^k$, $b_k = a_k r^k$, we have $\infty > \|f(rz)\|^2_{H_s(c_1)} \geqq c\sum_{n=0}^\infty (n+1)^{-s}|b_n|^2$ for some $c > 0$, according to [6]. For $|z| = \rho < 1$ we have

$$|f(rz)| \leqq \sum_{n=0}^\infty |b_n|\rho^n \leqq \sqrt{\sum_{n=0}^\infty |b_n|^2(n+1)^{-s}} \cdot \sqrt{\sum_{n=0}^\infty (n+1)^s\rho^{2n}}.$$

Differentiating the equality $1/(1-\rho) = \sum_{n=0}^\infty \rho^n$ k times, we get

$$\frac{k!}{(1-\rho)^{k+1}} = \sum_{n=0}^\infty (n+k)(n+k-1)\cdots(n+1)\rho^n.$$

Thus, for $k > s$, we have

$$\sqrt{\sum_{n=0}^\infty (n+1)^s\rho^{2n}} \leqq \sqrt{\sum_{n=0}^\infty (n+1)^k\rho^n} \leqq \sqrt{\sum_{n=0}^\infty (n+k)\cdots(n+1)\rho^n} = \sqrt{\frac{k!}{(1-\rho)^{k+1}}};$$

hence

$$|f(rz)| \leqq \|f(z)\|_{H_s^2(c_r)} \cdot \frac{\sqrt{k!}}{(1-\rho)^{(k+1)/2}} \quad (|z| \leqq \rho < 1),$$

$$|f(z)| \leqq \frac{c}{(r-\rho)^{(k+1)/2}} \quad (|z| \leqq \rho < r),$$

for some $c > 0$. Our statement is now a consequence of Theorem 4.

(4) Let G be a bounded and simply connected domain of the z-plane, with the boundary γ containing more than one point, and let γ_q, $q < 1$, be the image of the circle c_q under the conformal mapping $z = \psi(w)$ ($w = \phi(z)$ is the inverse mapping) of the domain G onto the unit disc $\operatorname{int}(c_1)$ in the w-plane.

THEOREM 6. *The function $f(z)$ belongs to $E^2_+(\gamma)$ if and only if $f(z) \in E^2_+(\gamma_q)$ and for every integer $n \geqq 0$ there exists a polynomial $p_n(w)$ of degree at most n such that*

$$(5) \qquad \|f(z) - p_n(\phi(z))\sqrt{\phi'(z)}\|_{E^2_+(\gamma_q)} \leqq q^n \cdot \beta_n \qquad (n = 0, 1, \cdots)$$

where $\{\beta_n\}_0^\infty \in l^2$.

PROOF. We consider the mapping $Tf(z) = f(\psi(w))\sqrt{\psi'(w)}$. From the equality

$$\|f(z)\|^2_{E^2_+(\gamma)} = \sup_n \int_{\gamma_{q_n}} |f(z)|^2 |dz| = \sup_n \int_{c_{q_n}} |f(\psi(w))|^2 |\sqrt{\psi'(w)}|^2 |dw|,$$

where $q_n \uparrow 1$, it follows that T is an isomorphic mapping of $E^2_+(\gamma)$ onto $E^2_+(c_1)$ and of $E^2_+(\gamma_q)$ onto $E^2_+(c_q)$. We have $T^{-1}w^k = [\phi(z)]^k \sqrt{\phi'(z)}$, so that our statement follows from Theorem 3.

REMARK. An analogous statement could be made in relation to the Rudin space ([5]):

$$H^2_+(\gamma) = \{f(z) : f(z) \in \mathcal{A}_+(\gamma), f(\psi(w)) \in H^2_+(c_1)\}$$

with $\|f(z)\|_{H^2_+(\gamma)} = \sup_n \int_{\gamma_{q_n}} |f(z)|^2 |\phi'(z)| |dz|$. In this case we would consider the mapping $Tf(z) = f(\psi(w))$ and $T^{-1}w^k = (\phi(z))^k$ so that instead of (5) we could have the condition

$$(6) \qquad \|f(z) - p_n(\phi(z)\|_{H^2_+(\gamma_q)} \leqq q^n \cdot \beta_n \qquad (n = 0, 1, 2, \cdots).$$

(5) Let K be a continuum in the z-plane and let $z = \psi(w) = w + \alpha_0 + \alpha_{-1}/w + \cdots$ (with the inverse mapping $w = \phi(z)$) be the conformal mapping of the component K_∞ of its complementary domain K^c which contains the point $z = \infty$, onto the exterior of the circle c_ρ. Let γ_r be the image of c_r,

$\rho < \leqq \infty$, under this mapping, and let $\{F_n(z)\}_{n=0}^{\infty}$ be the sequence of Faber's polynomials belonging to the continuum K ([3]). We know that the series $\Sigma_{n=0}^{\infty} a_n F_n(z)$ converges in the topology of $\mathscr{A}_+(\gamma_R)$ if and only if $\Sigma_{n=0}^{\infty} a_n w^n$ converges in $\mathscr{A}_+(c_R)$. We denote the following set by $l_+^2(\gamma_r)$:

$$l_+^2(\gamma_r) = \left\{ f(z) : f(z) = \sum_{n=0}^{\infty} a_n F_n(z), \| f(z) \|^2_{l_+^2(\gamma_r)} = \sum_{n=0}^{\infty} | a_n |^2 r^{2n} < \infty \right\}.$$

Now $l_+^2(\gamma_r) \subset \mathscr{A}_+(\gamma_r)$, $\rho < r < \infty$. The operator $Tf(z) = T \Sigma_{n=0}^{\infty} a_n F_n(z) = \Sigma_{n=0}^{\infty} a_n w^n$ maps the space $l_+^2(\gamma_r)$ onto $H_+^2(c_r)$ isomorphically, for every r, $\rho < r < \infty$. We observe that $T^{-1} w^n = F_n(z)$, $n = 0, 1, \cdots$.

THEOREM 7. *The function $f(z)$ belongs to $l_+^2(\gamma_R)$ if and only if $f(z) \in l_+^2(\gamma_r)$ for some r, $\rho < r < R$, and, for every integer $n \geqq 0$, there exists a polynomial $p_n(z)$ of degree at most n, such that $\| f(z) - p_n(z) \|_{l_+^2(\gamma_r)} \leqq (r/R)^n \beta_n$ $(n = 0, 1, 2, \cdots)$, where $\{\beta_k\}_0^{\infty} \in l^2$.*

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